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CONFERENCE

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Book of Abstracts

Dedicated to the 70th anniversary of Prof. Martin Grigoryan

Maximal functions and oscillatory integrals.

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In this mini-course, we explore extensions of seminal results in harmonic analysis, focusing on maximal functions associated with convex bodies and their generalizations to graded Lie groups. Building upon Bourgain's pioneering work, we delve into the analysis of maximal functions on graded Lie groups, formulating criteria based on the group Fourier transform. The course comprises three main sections:

Extending Bourgain's L^2-estimate to graded Lie groups, we investigate maximal functions associated with kernels possessing differentiable Fourier transforms. Our criterion, rooted in group Fourier analysis, yields insights into the L^p-boundedness of these maximal functions.

Drawing from Bourgain's insights on $\mathbb{R}^n$, we extend his results to graded Lie groups, particularly focusing on full maximal operators. Utilizing group Fourier transforms and positive left-invariant hypoelliptic partial differential operators, we articulate an extension applicable to these groups.

Expanding on the classical result by Calderón, Coifman, and Weiss, we establish analogs on graded Lie groups for dyadic maximal functions. By associating dyadic maximal functions with finite Borel measures on graded Lie groups, we present criteria for their boundedness on L^p spaces.

Introduction to hypercomplex function theory

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We will present an introduction to hypercomplex analysis, in particular, to its functional analytic and function theoretical foundations. As it is well known there is no function theory in higher dimensions which incorporates all tools from complex analysis. Here we will use an approach where the field of complex numbers is replaced by Clifford algebras. The main advantage of this approach consists in the possibility to combine much of the algebraical and geometrical properties of complex numbers while taking into account the geometric properties in higher dimensions, such as the non-commutativity of rotations. After a short introduction, we develop the necessary tools from a functional analytic and function theoretical point of view. We will show how these can be applied to study spectra of operators as well as in the study of reproducing kernel spaces and its application in white noise analysis.

Introduction to stratified Lie groups

Marianna Chatzakou
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The primary aim of these lectures will be to offer a comprehensive introduction to a particular class of nilpotent Lie groups, referred to as stratified (or Carnot) groups. A trivial example in this class is the Euclidean space $\mathbb{R}^n$. Apart from this trivial case, the Heisenberg group $\mathbb{H}^n$ is the most well-studied example of a stratified Lie group. In the first lecture, we will define the stratified groups, see how to derive their corresponding Lie algebras, and discuss the main properties of both. We will see what are the operations in both settings and how the differential operators on a stratified group are defined. In the second lecture, we will see what kind of difficulties one will face when tackling problems in analysis and geometry in the setting of stratified groups. In particular, we will discuss the questions that will arise when one tries to extend Euclidean results to the more general setting of a stratified group.

Microlocal analysis and Quantum Field Theory

Claudio Dappiaggi
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I will give a basic introduction to quantum field theory using the algebraic approach. The main goal is to show how and why microlocal analysis plays a distinguished róle among the mathematical structures used in a mathematically coherent formulation of field theories. These lectures are based partly on [1, 2].

Lecture 1

I shall introduce the basic ingredients at the heart of classical field theory by considering the basic example of a free, scalar field on Minkowski spacetime:

- Minkowski spacetime and its causal structure;
- The real Klein-Gordon field: kinematics and dynamics;
- Fundamental solutions of the Klein-Gordon operator (advanced, retarded and the Feynman propagators);
- The space of smooth, spacelike compact solutions of the Klein-Gordon Equation and its symplectic structure.

Lecture 2

I shall discuss the algebraic quantization of a Klein-Gordon field without assuming any a priori knowledge of quantum mechanics. In particular I will be focusing on the following topics:

- The algebra of classical linear observables;
- The Canonical Commutation Relations (CCRs) and the Algebra of Fields;

- The notion of algebraic states and the Gelfand-Naimark-Segal theorem;
- The n-point correlation functions and Gaussian states.

Lecture 3

In the last lecture I will show how and why microlocal analysis plays a distinguished róle among the mathematical structures at the heart of quantum field theory:

- Hadamard states: local form;
- Global Hadamard states and their wavefront set;
- Radzikowski theorem: Equivalence between the global and the local form;
- The Hadamard parametrix and Wick polynomials.

References

[1] M. Benini and C. Dappiaggi, “Models of free quantum field theories on curved backgrounds,” in R. Brunetti et al. “Advances in algebraic quantum field theory,” Springer, (2015), [arXiv:1505.04298 [math-ph]].

[2] I. Khavkine and V. Moretti, “Algebraic QFT in Curved Spacetime and quasifree Hadamard states: an introduction,” in R. Brunetti et al. “Advances in algebraic quantum field theory,” Springer, (2015), [arXiv:1412.5945 [math-ph]].

Calculus of Günter’s derivatives on hypersurfaces and Γ-convergence

Roland Duduchava
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Partial differential equations on Riemannian manifolds are usually written in intrinsic local coordinates, involving metric tensor and Christoffel symbols. But if we deal with a hypersurface, the Cartesian coordinates of the ambient space can be applied. We propose an approach which allows global representation of basic differential operators (such as Laplace-Beltrami, Hodge-Laplacian, Lamé, Navier-Stokes, Shell equations etc.) and of corresponding boundary value problems on a hypersurface $\mathcal{C} \subset \mathbb{R}^n$ with or without the boundary in terms of tangent “Günter’s and Stoke’s derivative”. The representation of operators is based on the normal vector field. The tools we develop also provide, in some important cases, useful simplifications as well as new interpretations of classical operators and equations.

This seemingly trivial idea simplifies the form of classical differential equations on the surface, mentioned above, which turn out to have constant coefficients, and enables more transparent proofs of Korn’s inequalities, tightly connected with solvability and uniqueness of some boundary value problems.

Another tool which will be discussed is the Γ-convergence. This tool allows rigorous derivation of limit equation of Mathematical Physics (minimization problem for solutions to BVPs, such as stationary heat transfer, Deformation of elastic material etc.) in thin layers when the thickness of the layer tends to 0.

Lecture 1: We will expose basics of calculus of tangent Günter's and Stoke's derivatives and show how classical differential operators, such as Laplace-Beltrami operator (governing the stationary heat conduction), Lamé-Beltrami operator (governing small deformations of an elastic material) and others are written in terms of Günter's derivatives.

Lecture 2: We will define Γ-convergence and discuss its properties, its usefulness in deriving boundary value problems for partial differential equations of Mathematical Physics in thin domains.

Lecture 3: We will end up with the demonstration of Γ-convergence for the stationary heat transfer equation in a thin layer and derive limit equation when the thickness tends to zero.

We will also discuss shortly the derivation of shell equations for thin layer by using Γ-convergence.

How to make use of Wiener Amalgam Spaces

Hans Georg Feichtinger
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This course will consist of three units of one hour each. The first two units will be presented via ZOOM (online) while the last one, taking place on Saturday 29th of June, will be done "live".

It is the purpose of this series of lectures to explain what Wiener amalgam spaces are, how to make use of their properties in various settings and to demonstrate that they are a universal tool for Gabor Analysis, but also for classical Fourier Analysis.

Basically, the Wiener amalgam spaces $W(B, C)$ describe the global behavior (typically expressed by some weighted L^q-spaces C) of a local property (which can be either another L^p-norm, or the Fourier algebra, or some Sobolev algebra). The typical starting point are so-called BUPUs (bounded uniform partitions of unity), but the equivalence between the corresponding discrete norms and the continuous ones is one of the starting points for this theory. Duality, pointwise multipliers and convolution results all can be derived in a "coordinatewise manner" (local by local and separately global by global). There is also a Hausdorff-Young type Theorem which allows to provide an easy proof of Sobolev's Embedding Theorem.

Modulation spaces have been introduced as the inverse image of such Wiener amalgam spaces, with FL^p (Fourier image of L^p) as a local component, similar to the Frazier-Jawerth characterization of Besov spaces, using dyadic partitions of unity. Towards the end of the course modulation spaces, in particular the unweighted ones, will be discussed in some detail. The smallest among the family of modulation spaces $M^{p,q}(R^d)$ is the so-called Feichtinger algebra $M^1(R^d)$, usually denoted by $S_0(R^d)$ and the largest, the dual of $S_0(R^d)$ (or also $M^\infty(R^d)$, the space of mild distributions.

Basic concepts for the theory of mild distributions (such as regularization via product-convolution operators or finite Gabor partial sums) are also derived with the help of basic properties of Wiener amalgam spaces (now over phase space, the so-called time-frequency plane).

It is the overall goal to demonstrate that it is not hard to understand how to work with Wiener amalgams, even if one is not familiar with the fine technical details, and that they provide a set of tools with far reaching applications. In this sense the course will explain why they get high priority in the author's informal ranking described in his paper "Choosing Function Spaces" from 2015.

Reproducing kernel Hilbert and Krein spaces in complex and hypercomplex analysis

Uwe Kähler
Department of Mathematics, University of Aveiro, Portugal

We discuss the notion of reproducing kernel Hilbert and Krein spaces in the context of complex and hypercomplex analysis, including basic questions like the existence and uniqueness of such spaces. We will study examples like Hardy, Bergman, and Fock space as well as their specific properties. In the end we also discuss applications of such spaces in different areas, from interpolation to machine learning.

Very weak solutions of singular PDEs (FAST Lectures[1])

Michael Ruzhansky
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This minicourse will cover very weak solutions of some partial differential equations with singular coefficients, including the basic concepts and examples. The well-posedness of the Cauchy problem for some very classical partial differential equations with variable coefficients, such as the wave equation or the heat equation in inhomogeneous media, is traditionally established by so-called energy methods. When the coefficients of the equation are singular, which means, that the qualities of the medium change abruptly, not only do these methods fail, but also the very meaning in which a weak solution may satisfy the equation becomes elusive. The framework of very weak solutions allows to address such questions and establish a kind of well-posedness for equations with singular coefficients.

In this course, we will first demonstrate the energy methods for very basic equations with variable coefficients. Then we will show how things fail when one of the coefficients is singular. Then we will introduce the framework of very weak solutions, and show how the well-posedness can be established in this new context.

[1]This series of lectures is presented in the framework of the ADVANCE programme by FAST Foundation. https://fast.foundation/

Qualitative properties of solutions for parabolic PDEs

Armen Shirikyan
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The goal of this minicourse is to discuss some basic results on the existence and uniqueness of solutions of parabolic PDEs in Hölder spaces and to study qualitative properties of those solutions. To simplify the presentation, we confine ourselves to the one-dimensional case and impose periodic boundary conditions. We first establish the existence and uniqueness of solutions in an appropriate functional class in the linear case and show how to use that result to handle the nonlinear case. We next turn to some qualitative properties of solutions for linear equations, such as the maximum principle, its dual property of contraction of the L^1-norm, and the Harnack inequality. Finally, in the nonlinear case, we establish a property of strong nonlinear dissipation and discuss its application in the theory of control and stabilisation.

Plan:

1. Initial-boundary value problem for linear and nonlinear equations
2. Maximum principle and Harnack inequality
3. Nonlinear dissipation
4. Applications in the control theory of the Burgers equation

On twisted Heisenberg groups

Sundaram Thangavelu
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As is well known, the translations $\tau(a), a \in \mathbb{R}^n$ and modulations $e(b), b \in \mathbb{R}^n$ acting as unitary operators on $L^2(\mathbb{R}^n)$ give rise to the Heisenberg group $\mathbb{H}^n$. Now the Heisenberg group induces, for each non-zero real λ, the twisted translations $\tau_\lambda(a,b), (a,b) \in \mathbb{H}^{2n}$ acting on $L^2(\mathbb{R}^{2n})$ as unitary operators. Using symplectic Fourier transform on $L^2(\mathbb{R}^{2n})$ we can define the twisted modulations $e_\lambda(a',b'), (a',b') \in \mathbb{R}^{2n}$. These two families give rise to a group which we call the twisted Heisenberg group. For this group we describe an irreducible unitary representation ρ_λ realised on Hilbert space of entire functions called twisted Fock spaces.

Spectral theory of differential operators: what's it all about and what is its use

Dmitri Vassiliev
Department of Mathematics, University College London, United Kingdom

I will give an overview of the spectral theory of partial differential operators, charting its development from the non-rigorous works of physicists to modern rigorous mathematical results.

Lecture 1

Basic examples of spectral problems: pendulum, mass on a spring, mass on several springs, vibrating string, membrane, acoustic resonator and electromagnetic resonator.

Eigenvalue counting function.

The Rayleigh–Jeans formula.

Weyl's proof of the one-term asymptotic formula for the counting function of the Laplacian. Courant's proof.

Weyl's conjecture on the existence of a two-term asymptotic formula for the counting function of the Laplacian. Issues associated with analytic number theory.

Contributions from Levitan, Fedosov, Kuznetsov, Hörmander, Duistermaat, Guillemin and Melrose. The wave equation method and Fourier Tauberian theorems.

Ivrii's proof of Weyl's conjecture. Geometric conditions on the billiard flow.

Lecture 2

Two-term asymptotics for higher order operators. This lecture is based on the book [1].

Lecture 3

Two-term asymptotics for systems. I will consider systems of two types.

Type 1 Self-adjoint elliptic system of m PDEs, each PDE of even order $2n$, on a compact d-dimensional manifold M with boundary ∂M. Requires mn boundary conditions. The system is assumed to be semi-bounded from below. Here the exposition is based on the paper [3].

Type 2 Self-adjoint elliptic system of m PDEs, each PDE first order, on a compact d-dimensional manifold M without boundary. Semi-boundedness is not assumed. Think particle/antiparticle. Here the exposition is based on the paper [2].

In the end of the lecture I will briefly discuss some real-life applications of spectral theory.

References

[1] Yu. Safarov and D. Vassiliev, The asymptotic distribution of eigenvalues of partial differential operators, Amer. Math. Soc., Providence (RI), 1997.

[2] O. Chervova, R. J. Downes and D. Vassiliev, The spectral function of a first order elliptic system, J. Spectr. Theory 3 no. 3 (2013) 317-360.

[3] M. Capoferri, L. Friedlander, M. Levitin and D. Vassiliev, Two-term spectral asymptotics in linear elasticity, J. Geom. Anal. 33 (2023), 242. DOI:10.1007/s12220-023-01269-y.

Nonlinear generalized functions and nonstandard principles

Hans Vernaeve
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We give an introduction to nonlinear generalized functions. They arise from the need to study PDE without solutions in the sense of Schwartz distributions, often because in the equation, ill-defined operations on distributions are present. In this case, one defines a so-called very weak solution as a net of solutions to the PDE with regularized data. We indicate their relation with Schwartz distributions and discuss how one can develop a theory of these objects based upon principles originating from nonstandard analysis. The latter is a theory that allows to rigorously use infinitesimals in mathematical analysis. Nets of solutions to a PDE have arisen in this theory as models for solutions to the PDE taking values in an extension of the real number line with infinitesimals. Although not identical to the solutions in the nonstandard analysis framework, a number of principles from nonstandard analysis can still be used to study nonlinear generalized functions.